Practice Second Midterm Exam III

This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 48 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

Question	Points	Grader
(1) Discrete Structures I	/ 12	
(2) Discrete Structures II	/ 12	
(3) Discrete Structures III	/ 12	
(4) Discrete Structures IV	/ 12	
	/ 48	

<u>You can do this.</u> Best of luck on the exam!

Problem One: Discrete Structures I (12 Points)

(Midterm Exam, Spring 2017)

On Problem Set Four, you explored how set cardinality relates back to bijective functions between sets. This problem is designed to let you show us what you've learned in the course of working through those problems.

Consider the following two sets:

$$\mathbb{E} = \{\ n \in \mathbb{Z} \mid n \text{ is even}\ \}$$
$$\mathbb{O} = \{\ n \in \mathbb{Z} \mid n \text{ is odd}\ \}$$

Notice that these are sets of *integers*, not *natural numbers*.

In this problem, we're going to ask you to prove that $|\mathbb{E}| = |\mathbb{O}|$.

i. **(2 Points)** Draw a picture showing a way to pair off the elements of $\mathbb{E}$ with the elements of $\mathbb{O}$.

ii. **(2 Points)** Based on the picture you came up with in part (i), define a bijection $f : \mathbb{E} \to \mathbb{O}$. (You'll prove that your function is a bijection in part (iii). We're just expecting the definition of f here.)

iii. **(8 Points)** Prove that $|\mathbb{E}| = |\mathbb{O}|$ by proving the function you came up with in part (ii) is a bijection. In the course of writing up your proof, please briefly prove that the function you've chosen is well-defined (that is, every input in the domain produces an output in the codomain.)

Problem Two: Discrete Structures II (12 Points)

(Midterm Exam, Winter 2018)

On Problem Set Three, Problem Set Four, and Problem Set Five, you explored different properties of strict order relations and of functions between sets. This problem is designed to let you show us what you've learned about these types of structures.

Let's begin with a new definition. If R_1 is a binary relation over a set A_1 and R_2 is a binary relation over a set A_2, then an ***embedding of R_1 in R_2*** is a function $f : A_1 \to A_2$ such that

$$\forall a \in A_1.\ \forall b \in A_1.\ (aR_1b \leftrightarrow f(a)\ R_2\ f(b)).$$

If there's an embedding of a relation R_1 in a relation R_2, we say that R_1 ***can be embedded in*** R_2.

i. **(6 Points)** Let R_1 be a binary relation over a set A_1 and let R_2 be a *strict order* over some set A_2. Prove that if R_1 can be embedded in R_2, then R_1 is a strict order.

   Feel free to use the space below for scratch work. There's room to write your answer to this problem on the next page of this exam.

(Extra space for your answer to Problem Two, Part (i), if you need it.)

As a refresher, if R_1 is a binary relation over a set A_1 and R_2 is a binary relation over a set A_2, then an ***embedding of R_1 in R_2*** is a function $f : A_1 \to A_2$ such that

$$\forall a \in A_1.\ \forall b \in A_1.\ (aR_1b \leftrightarrow f(a)\ R_2\ f(b)).$$

If there's an embedding of a relation R_1 in a relation R_2, we say that R_1 ***can be embedded in*** R_2.

ii. **(6 Points)** In the indicated space below, draw diagrams of two binary relations R_1 and R_2 (which may or may not be over the same set) and an embedding f of R_1 in R_2 such that

- R_1 is a strict order, but
- R_2 is ***not*** a strict order.

Then, briefly justify your answer. There's space on the next page of this exam for scratch work.

Draw relation R_1 here.

Draw relation R_2 here.

Draw the embedding $f : A_1 \to A_2$ by drawing arrows from each element of the domain to its corresponding element of the codomain.

Briefly justify your answer here:

(Scratch space for your answer to Problem Two, Part (ii), if you need it.)

Problem Three: Discrete Structures III (12 Points)

(Midterm Exam, Spring 2017)

On Problem Set Four, you explored bipartite graphs and independent sets. This problem explores a generalization of bipartite graphs and its connection to independent sets.

A graph $G = (V, E)$ is called a ***k-partite graph*** if there are k sets $V_1, V_2, \ldots, V_k$ such that

- every node $v \in V$ belongs to exactly one set V_i, and
- every edge $e \in E$ has its endpoints in two different sets V_i and V_j.

These sets are called the ***k-partite classes*** of G.

Now, a quick refresher. An ***independent set*** in a set $G = (V, E)$ is a set $I \subseteq V$ where

$$\forall u \in I. \; \forall v \in I. \; \{u, v\} \notin E.$$

Let $G = (V, E)$ be a k-partite graph for some natural number $k \geq 2$. Prove that if G has exactly n nodes, then G has an independent set of size at least $\lceil n/k \rceil$.

(Extra space for your answer to Problem Three, if you need it.)

Problem Four: Discrete Structures IV (12 Points)

(Midterm Exam, Spring 2017)

Let $f : A \to A$ be an arbitrary function from a set A to itself. We can inductively define a class of functions f^0, f^1, f^2, etc. called the ***powers of f*** as follows:

$$\begin{aligned} f^0(x) &= x \\ f^{n+1}(x) &= (f \circ f^n)(x) \end{aligned}$$

This question explores properties of powers of functions.

i. **(3 Points)** Let $f : A \to A$ be an arbitrary function. Prove that $f^1(x) = f(x)$ for all $x \in A$.

As a refresher, if $f : A \to A$, then we can inductively define a class of functions f^0, f^1, f^2, etc. called the ***powers of f*** as follows:

$$f^0(x) = x$$
$$f^{n+1}(x) = (f \circ f^n)(x)$$

Now, a new definition. A ***fixed point*** of a function $f : A \to A$ is an element $a \in A$ such that $f(a) = a$.

ii. **(9 Points)** Let $f : A \to A$ be an arbitrary function. Your task is to prove the following statement:

For any natural number n, and for any element $a \in A$,
if a is a fixed point of f, then a is a fixed point of f^n.

To do so, we'd like you to use induction. Specifically, use induction to prove that the statement $P(n)$ defined below is true for all natural numbers n:

$P(n)$ is the statement "for any $a \in A$, if a is a fixed point of f, then a is a fixed point of f^n."

(Extra space for your answer to Problem Four, Part (ii), if you need it.)